3 ANALYSIS AND TRANSMISSION OF SIGNALS

Electrical engineers instinctively think of signals in terms of their frequency spectra and of systems in terms of their frequency responses. Even teenagers know about audio signals having a bandwidth of 20 kHz and good-quality loudspeakers responding up to 20 kHz. This is basically thinking in the frequency domain. In the last chapter we discussed spectral representation of periodic signals (Fourier series). In this chapter we extend this spectral representation to aperiodic signals.

3.1 APERIODIC SIGNAL REPRESENTATION BY FOURIER INTEGRAL

We now apply a limiting process to show that an aperiodic signal can be expressed as a continuous sum (integral) of everlasting exponentials. To represent an aperiodic signal $g(t)$ such as the one shown in Fig. 3.1a by everlasting exponential signals, let us construct a new periodic signal $g_{T_0}(t)$ formed by repeating the signal $g(t)$ every T_0 seconds, as shown in Fig. 3.1b. The period T_0 is made long enough to avoid overlap between the repeating pulses. The periodic signal $g_{T_0}(t)$ can be represented by an exponential Fourier series. If we let $T_0 \to \infty$, the pulses in the periodic signal repeat after an infinite interval, and therefore

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)$$

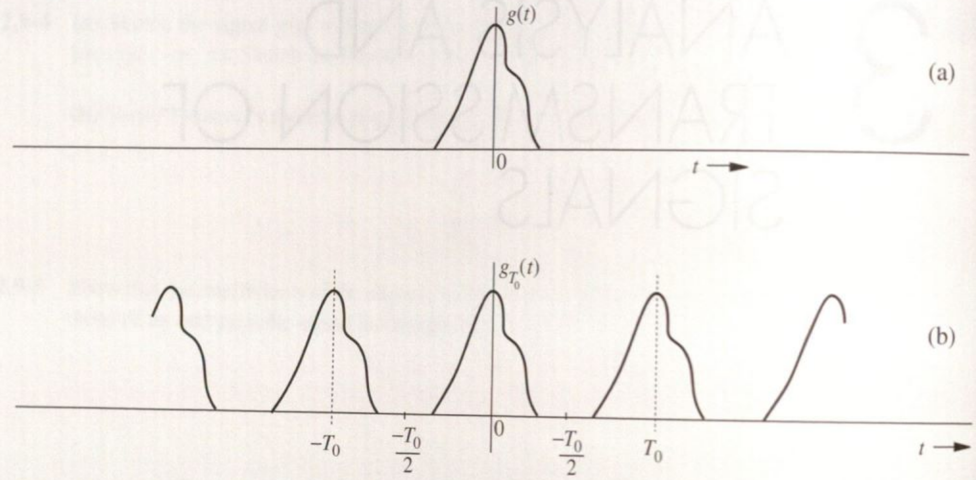
Thus, the Fourier series representing $g_{T_0}(t)$ will also represent $g(t)$ in the limit $T_0 \to \infty$. The exponential Fourier series for $g_{T_0}(t)$ is given by

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{3.1}$$

in which

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-jn\omega_0 t}\, dt \tag{3.2a}$$

Figure 3.1
Construction of a periodic signal by periodic extension of $g(t)$.



and

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0 \tag{3.2b}$$

Observe that integrating $g_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. Therefore, Eq. (3.2a) can be expressed as

$$\begin{aligned} D_n &= \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t}\,dt \qquad (3.2c) \\ &= \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-j2\pi n f_0 t}\,dt \end{aligned}$$

It is interesting to see how the nature of the spectrum changes as T_0 increases. To understand this fascinating behavior, let us define $G(f)$, a continuous function of ω, as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt \tag{3.3}$$

$$= \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\,dt \tag{3.4}$$

in which $\omega = 2\pi f$. A glance at Eqs. (3.2c) and (3.3) then shows that

$$D_n = \frac{1}{T_0}G(nf_0) \tag{3.5}$$

This in turn shows that the Fourier coefficients D_n are ($1/T_0$ times) the samples of $G(f)$ uniformly spaced at intervals of f_0 Hz, as shown in Fig. 3.2a.*

Therefore, $(1/T_0)G(f)$ is the envelope for the coefficients D_n. We now let $T_0 \to \infty$ by doubling T_0 repeatedly. Doubling T_0 halves the fundamental frequency f_0, so that there are now twice as many components (samples) in the spectrum. However, by doubling T_0, we

* For the sake of simplicity we assume D_n and therefore $G(f)$ in Fig. 3.2 to be real. The argument, however, is also valid for complex D_n [or $G(f)$].

Figure 3.2
Change in the Fourier spectrum when the period T_0 in Fig. 3.1 is doubled.

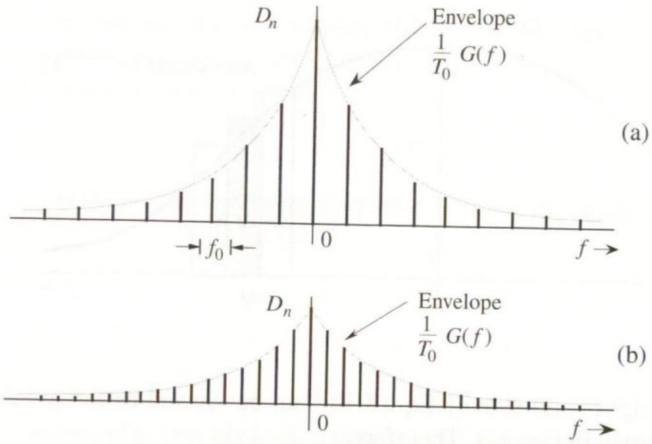


Figure 3.2 Change in the Fourier spectrum when the period T_0 in Fig. 3.1 is doubled.

have halved the envelope $(1/T_0)\,G(f)$, as shown in Fig. 3.2b. If we continue this process of doubling T_0 repeatedly, the spectrum progressively becomes denser while its magnitude becomes smaller. Note, however, that the relative shape of the envelope remains the same [proportional to $G(f)$ in Eq. (3.3)]. In the limit as $T_0 \rightarrow \infty, f_0 \rightarrow 0$ and $D_n \rightarrow 0$. This means that the spectrum is so dense that the spectral components are spaced at zero (infinitesimal) interval. At the same time, the amplitude of each component is zero (infinitesimal). We have *nothing of everything, yet we have something!* This sounds like *Alice in Wonderland*, but as we shall see, these are the classic characteristics of a very familiar phenomenon.*

Substitution of Eq. (3.5) in Eq. (3.1) yields

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(nf_0)}{T_0} e^{jn2\pi f_0 t} \tag{3.6}$$

As $T_0 \rightarrow \infty, f_0 = 1/T_0$ becomes infinitesimal ($f_0 \rightarrow 0$). Because of this, we shall use a more appropriate notation, Δf, to replace f_0. In terms of this new notation, Eq. (3.2b) becomes

$$\Delta f = \frac{1}{T_0}$$

and Eq. (3.6) becomes

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} [G(n\Delta f)\Delta f]\, e^{(j2\pi n\Delta f)t} \tag{3.7a}$$

Equation (3.7a) shows that $g_{T_0}(t)$ can be expressed as a sum of everlasting exponentials of frequencies $0, \pm\Delta f, \pm 2\Delta f, \pm 3\Delta f, \ldots$ (the Fourier series). The amount of the component of frequency $n\Delta f$ is $[G(n\Delta f)\Delta f]$. In the limit as $T_0 \rightarrow \infty$, $\Delta f \rightarrow 0$ and $g_{T_0}(t) \rightarrow g(t)$. Therefore,

$$g(t) = \lim_{T_0 \rightarrow \infty} g_{T_0}(t) = \lim_{\Delta f \rightarrow 0} \sum_{n=-\infty}^{\infty} G(n\Delta f) e^{(j2\pi n\Delta f)t}\, \Delta f \tag{3.7b}$$

* You may consider this as an irrefutable proof of the proposition that 0% ownership of everything is better than 100% ownership of nothing!

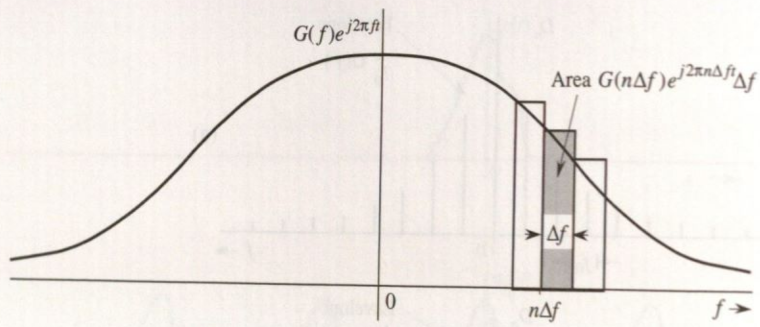


Figure 3.3 The Fourier series becomes the Fourier integral in the limit as $T_0 \rightarrow \infty$.

The sum on the right-hand side of Eq. (3.7b) can be viewed as the area under the function $G(f)e^{j2\pi ft}$, as shown in Fig. 3.3. Therefore,

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df \tag{3.8}$$

The integral on the right-hand side is called the **Fourier integral**. We have now succeeded in representing an aperiodic signal $g(t)$ by a Fourier integral* (rather than a Fourier series). This integral is basically a Fourier series (in the limit) with fundamental frequency $\Delta f \rightarrow 0$, as seen from Eq. (3.7b). The amount of the exponential $e^{jn\Delta\omega t}$ is $G(n\Delta f)\Delta f$. Thus, the function $G(f)$ given by Eq. (3.4) acts as a spectral function.

We call $G(f)$ the **direct** Fourier transform of $g(t)$ and $g(t)$ the **inverse** Fourier transform of $G(f)$. The same information is conveyed by the statement that $g(t)$ and $G(f)$ are a Fourier transform pair. Symbolically, this is expressed as

$$G(f) = \mathcal{F}[g(t)] \quad \text{and} \quad g(t) = \mathcal{F}^{-1}[G(f)]$$

or

$$g(t) \Longleftrightarrow G(f)$$

To recapitulate,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt \tag{3.9a}$$

and

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df \tag{3.9b}$$

It is helpful to keep in mind that the Fourier integral in Eq. (3.9b) is of the nature of a Fourier series with fundamental frequency Δf approaching zero [Eq. (3.7b)]. Therefore, most

* This should not be considered to be a rigorous proof of Eq. (3.8). The situation is not as simple as we have made it appear.[1]

of the discussion and properties of Fourier series apply to the Fourier transform as well. We can plot the spectrum $G(f)$ as a function of f. Since $G(f)$ is complex, we have both amplitude and angle (or phase) spectra:

$$G(f) = |G(f)|e^{j\theta_g(f)}$$

in which $|G(f)|$ is the amplitude and $\theta_g(f)$ is the angle (or phase) of $G(f)$. From Eq. (3.9a),

$$G(-f) = \int_{-\infty}^{\infty} g(t)e^{j2\pi ft}\, dt$$

f versus ω

Traditionally, two equivalent notations of angular frequency ω and frequency f are often used in representing signals in the frequency domain. There is no conceptual difference between the use of angular frequency ω (in radians per second, rad/s) and frequency f (in hertz, Hz). Because of their direct relationship, we can simply substitute $\omega = 2\pi f$ into $G(f)$ to arrive at the Fourier transform relationship in the ω-domain:

$$\mathcal{F}[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\, dt \tag{3.10}$$

Because of the additional 2π factor in the variable ω used by Eq. (3.10), the inverse transform as a function of ω requires an extra division by 2π. Therefore, the notation of f is slightly favored in practice when we write Fourier transforms. For this reason, we shall, for the most part, denote the Fourier transforms of signals as functions of $G(f)$. On the other hand, the notation of angular frequency ω can also offer some convenience when we are dealing with sinusoids. Thus, in later chapters, whenever it is *convenient and nonconfusing,* we shall use the two equivalent notations interchangeably.

Conjugate Symmetry Property

From Eq. (3.9a), it follows that if $g(t)$ is a real function of t, then $G(f)$ and $G(-f)$ are complex conjugates, that is,*

$$G(-f) = G^*(f) \tag{3.11}$$

Therefore,

$$|G(-f)| = |G(f)| \tag{3.12a}$$

$$\theta_g(-f) = -\theta_g(f) \tag{3.12b}$$

Thus, for real $g(t)$, the amplitude spectrum $|G(f)|$ is an even function, and the phase spectrum $\theta_g(f)$ is an odd function of f. This property (the **conjugate symmetry property**) is valid only for real $g(t)$. These results were derived earlier for the Fourier spectrum of a periodic signal in Chapter 2 and should come as no surprise. *The transform $G(f)$ is the frequency domain specification of $g(t)$.*

---

* *Hermitian symmetry* is the term used to describe complex functions that satisfy Eq. (3.11)

Example 3.1 Find the Fourier transform of $e^{-at}u(t)$.

By definition [Eq. (3.9a)],

$$G(f) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j2\pi ft}\, dt = \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt = \frac{-1}{a+j2\pi f} e^{-(a+j2\pi f)t}\Big|_0^{\infty}$$

But $|e^{-j2\pi ft}| = 1$. Therefore, as $t \to \infty$, $e^{-(a+j2\pi f)t} = e^{-at}e^{-j2\pi ft} = 0$ if $a > 0$. Therefore,

$$G(f) = \frac{1}{a + j\omega} \qquad a > 0 \tag{3.13a}$$

where $\omega = 2\pi f$. Expressing $a + j\omega$ in the polar form as $\sqrt{a^2+\omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}$, we obtain

$$G(f) = \frac{1}{\sqrt{a^2+(2\pi f)^2}} e^{-j\tan^{-1}(2\pi f/a)} \tag{3.13b}$$

Therefore,

$$|G(f)| = \frac{1}{\sqrt{a^2+(2\pi f)^2}} \quad \text{and} \quad \theta_g(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

The amplitude spectrum $|G(f)|$ and the phase spectrum $\theta_g(f)$ are shown in Fig. 3.4b. Observe that $|G(f)|$ is an even function of f, and $\theta_g(f)$ is an odd function of f, as expected.

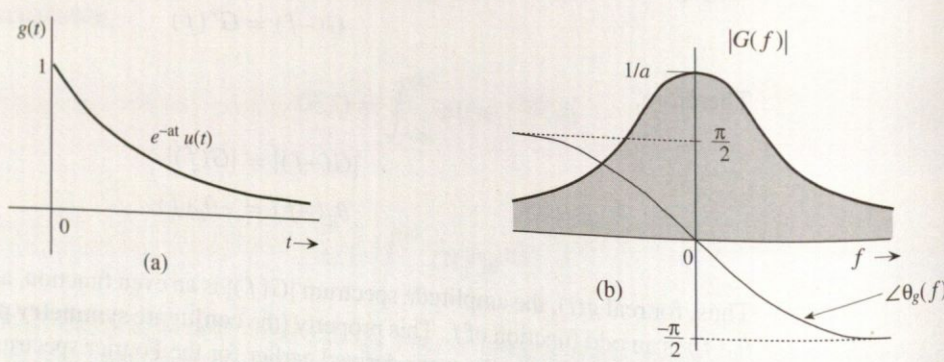


Figure 3.4 (a) $e^{-at}u(t)$ and (b) its Fourier spectra.

Existence of the Fourier Transform

In Example 3.1 we observed that when $a < 0$, the Fourier integral for $e^{-at}u(t)$ does not converge. Hence, the Fourier transform for $e^{-at}u(t)$ does not exist if $a < 0$ (growing exponential).

Clearly, not all signals are Fourier transformable. The existence of the Fourier transform is assured for any $g(t)$ satisfying the Dirichlet conditions given in Eq. (2.84). The first of these conditions is*

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty \tag{3.14}$$

To show this, recall that $|e^{-j2\pi ft}| = 1$. Hence, from Eq. (3.9a) we obtain

$$|G(f)| \leq \int_{-\infty}^{\infty} |g(t)|\, dt$$

This shows that the existence of the Fourier transform is assured if condition (3.14) is satisfied. Otherwise, there is no guarantee. We saw in Example 3.1 that for an exponentially growing signal (which violates this condition) the Fourier transform does not exist. Although this condition is sufficient, it is not necessary for the existence of the Fourier transform of a signal. For example, the signal $(\sin at)/t$, violates condition (3.14) but does have a Fourier transform. Any signal that can be generated in practice satisfies the Dirichlet conditions and therefore has a Fourier transform. Thus, the physical existence of a signal is a sufficient condition for the existence of its transform.

Linearity of the Fourier Transform (Superposition Theorem)

The Fourier transform is linear; that is, if

$$g_1(t) \Longleftrightarrow G_1(f) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(f)$$

then for all constants a_1 and a_2, we have

$$a_1 g_1(t) + a_2 g_2(t) \Longleftrightarrow a_1 G_1(f) + a_2 G_2(f) \tag{3.15}$$

The proof is trivial and follows directly from Eq. (3.9a). This theorem simply states that linear combinations of signals in the time domain correspond to linear combinations of their Fourier transforms in the frequency domain. This result can be extended to any finite number of terms as

$$\sum_k a_k g_k(t) \Longleftrightarrow \sum_k a_k G_k(f)$$

for any constants $\{a_k\}$ and signals $\{g_k(t)\}$.

Physical Appreciation of the Fourier Transform

To understand any aspect of the Fourier transform, we should remember that Fourier representation is a way of expressing a signal in terms of everlasting sinusoids, or exponentials. The Fourier spectrum of a signal indicates the relative amplitudes and phases of the sinusoids that are required to synthesize that signal. A periodic signal's Fourier spectrum has finite amplitudes and exists at discrete frequencies (f_0 and its multiples). Such a spectrum is easy

* The remaining Dirichlet conditions are as follows. In any finite interval, $g(t)$ may have only a finite number of maxima and minima and a finite number of finite discontinuities. When these conditions are satisfied, the Fourier integral on the right-hand side of Eq. (3.9b) converges to $g(t)$ at all points where $g(t)$ is continuous and converges to the average of the right-hand and left-hand limits of $g(t)$ at points where $g(t)$ is discontinuous.

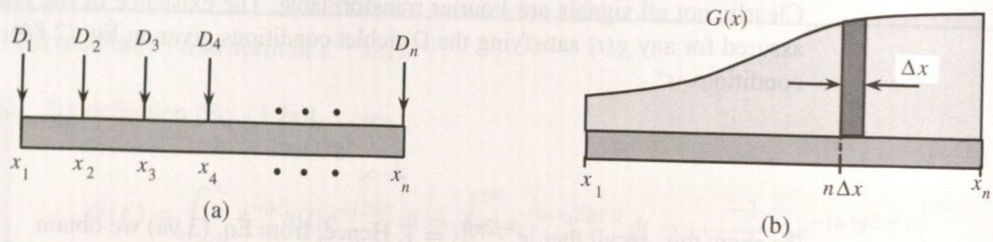


Figure 3.5 Analogy for Fourier transform.

to visualize, but the spectrum of an aperiodic signal is not easy to visualize because it has a continuous spectrum that exists at every frequency. The continuous spectrum concept can be appreciated by considering an analogous, more tangible phenomenon. One familiar example of a continuous distribution is the loading of a beam. Consider a beam loaded with weights $D_1, D_2, D_3, \ldots, D_n$ units at the uniformly spaced points $x_1, x_2, \ldots, x_n$, as shown in Fig. 3.5a. The total load W_T on the beam is given by the sum of these loads at each of the n points:

$$W_T = \sum_{i=1}^{n} D_i$$

Consider now the case of a continuously loaded beam, as shown in Fig. 3.5b. In this case, although there appears to be a load at every point, the load at any one point is zero. This does not mean that there is no load on the beam. A meaningful measure of load in this situation is not the load at a point, but rather the loading density per unit length at that point. Let $G(x)$ be the loading density per unit length of beam. This means that the load over a beam length Δx ($\Delta x \to 0$) at some point x is $G(x)\Delta x$. To find the total load on the beam, we divide the beam into segments of interval Δx ($\Delta x \to 0$). The load over the nth such segment of length Δx is $[G(n\Delta x)]\,\Delta x$. The total load W_T is given by

$$\begin{aligned} W_T &= \lim_{\Delta x \to 0} \sum_{x_1}^{x_n} G(n\Delta x)\,\Delta x \\ &= \int_{x_1}^{x_n} G(x)\,dx \end{aligned}$$

In the case of discrete loading (Fig. 3.5a), the load exists only at the n discrete points. At other points there is no load. On the other hand, in the continuously loaded case, the load exists at every point, but at any specific point x the load is zero. The load over a small interval Δx, however, is $[G(n\Delta x)]\,\Delta x$ (Fig. 3.5b). Thus, even though the load at a point x is zero, the relative load at that point is $G(x)$.

An exactly analogous situation exists in the case of a signal spectrum. When $g(t)$ is periodic, the spectrum is discrete, and $g(t)$ can be expressed as a sum of discrete exponentials with finite amplitudes:

$$g(t) = \sum_{n} D_n e^{j2\pi n f_0 t}$$

For an aperiodic signal, the spectrum becomes continuous; that is, the spectrum exists for every value of f, but the amplitude of each component in the spectrum is zero. The meaningful measure here is not the amplitude of a component of some frequency but the spectral density per unit bandwidth. From Eq. (3.7b) it is clear that $g(t)$ is synthesized by adding exponentials

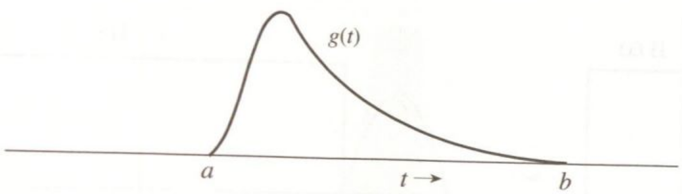


Figure 3.6 Time-limited pulse.

of the form $e^{j2\pi n\Delta f t}$, in which the contribution of any one exponential component is zero. But the contribution of exponentials in an infinitesimal band Δf located at $f = n\Delta f$ is $G(n\Delta f)\Delta f$, and the addition of all these components yields $g(t)$ in integral form:

$$g(t) = \lim_{\Delta f \to 0} \sum_{n=-\infty}^{\infty} G(n\Delta f) e^{(jn2\pi f)t} \Delta f = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df$$

The contribution of components within the band df is $G(f)\,df$, in which df is the bandwidth in hertz. Clearly $G(f)$ is the **spectral density** per unit bandwidth (in hertz). This also means that even if the amplitude of any one component is zero, the relative amount of a component of frequency f is $G(f)$. Although $G(f)$ is a spectral density, in practice it is customarily called the **spectrum** of $g(t)$ rather than the spectral density of $g(t)$. Deferring to this convention, we shall call $G(f)$ the Fourier spectrum (or Fourier transform) of $g(t)$.

A Marvelous Balancing Act

An important point to remember here is that $g(t)$ is represented (or synthesized) by exponentials or sinusoids that are everlasting (not causal). This leads to a rather fascinating picture when we try to visualize the synthesis of a time-limited pulse signal $g(t)$ (Fig. 3.6) according to the sinusoidal components in its Fourier spectrum. The signal $g(t)$ exists only over an interval (a, b) and is zero outside this interval. The spectrum of $g(t)$ contains an infinite number of exponentials (or sinusoids), which start at $t = -\infty$ and continue forever. The amplitudes and phases of these components are such that they add up exactly to $g(t)$ over the finite interval (a, b) and add up to zero everywhere outside this interval. Juggling with such a perfect and delicate balance of amplitudes and phases of an infinite number of components boggles the human imagination. Yet, the Fourier transform accomplishes it routinely, without much thinking on our part. Indeed, we become so involved in mathematical manipulations that we fail to notice this marvel.

3.2 TRANSFORMS OF SOME USEFUL FUNCTIONS

For convenience, we now introduce a compact notation for some useful functions such as rectangular, triangular, and interpolation functions.

Unit Rectangular Function

We use the pictorial notation $\Pi(x)$ for a rectangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.7a:

$$\Pi(x) = \begin{cases} 1 & |x| \le \dfrac{1}{2} \\ 0.5 & |x| = \dfrac{1}{2} \\ 0 & |x| > \dfrac{1}{2} \end{cases} \tag{3.16}$$

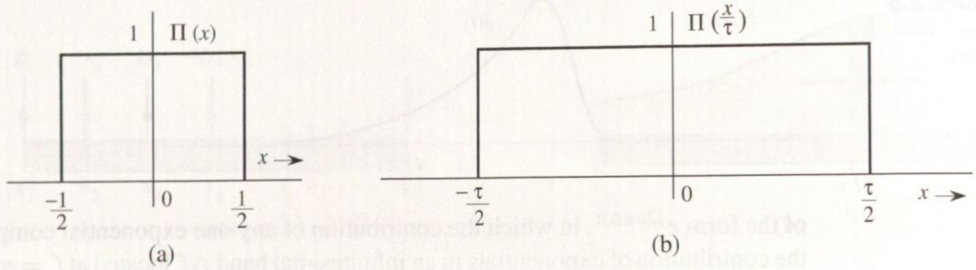


Figure 3.7 Rectangular pulse.

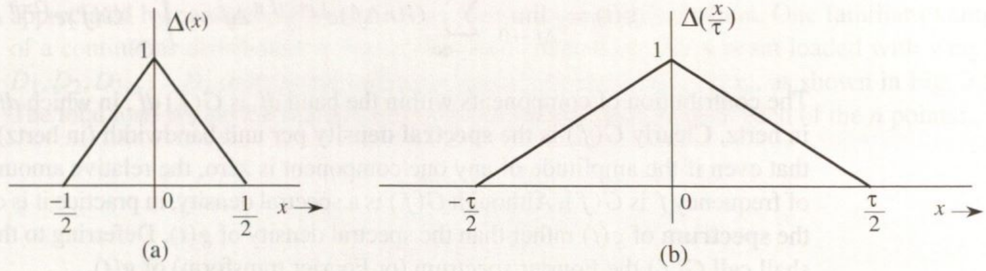


Figure 3.8 Triangular pulse.

Notice that the rectangular pulse in Fig. 3.7b is the unit rectangular pulse $\Pi(x)$ expanded by a factor τ and therefore can be expressed as $\Pi(x/\tau)$ (see Chapter 2). Observe that the denominator τ in $\Pi(x/\tau)$ indicates the width of the pulse.

Unit Triangular Function

We use a pictorial notation $\Delta(x)$ for a triangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.8a:

$$\Delta(x) = \begin{cases} 1 - 2|x| & |x| < \dfrac{1}{2} \\ 0 & |x| > \dfrac{1}{2} \end{cases} \tag{3.17}$$

Observe that the pulse in Fig. 3.8b is $\Delta(x/\tau)$. Observe that here, as for the rectangular pulse, the denominator τ in $\Delta(x/\tau)$ indicates the pulse width.

Sinc Function sinc (x)

The function $\sin x/x$ is the "sine over argument" function denoted by sinc (x).*

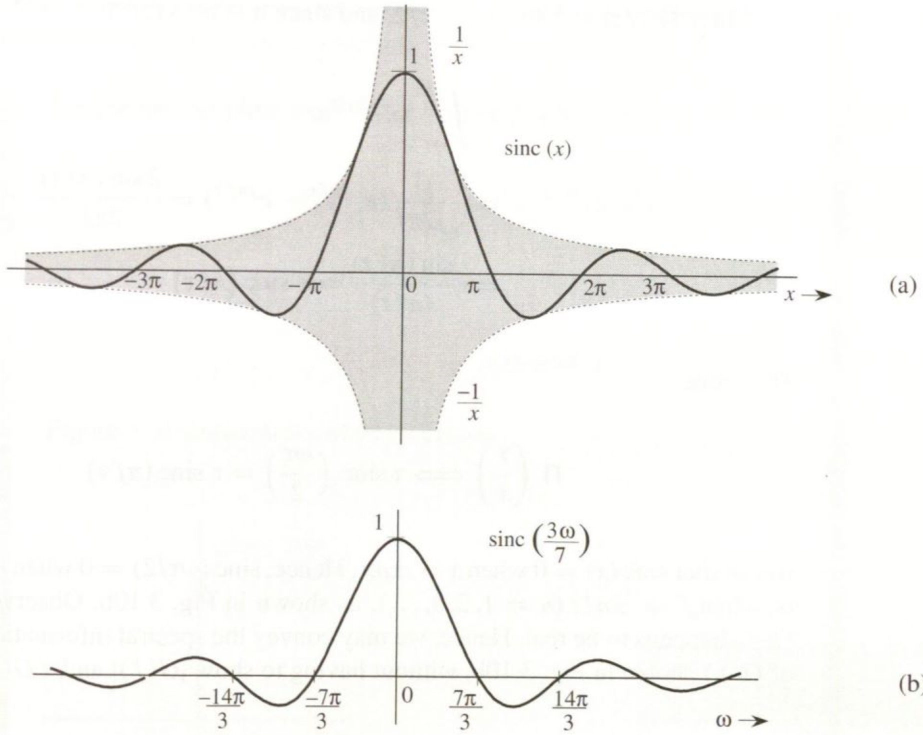
This function plays an important role in signal processing. We define

$$\text{sinc}\,(x) = \frac{\sin x}{x} \tag{3.18}$$

* The function sinc (x) is also denoted by Sa (x) in the literature. Some authors define sinc (x) as

$$\text{sinc}\,(x) = \frac{\sin \pi x}{\pi x}$$

Figure 3.9
Sinc pulse.



Inspection of Eq. (3.18) shows that

1. sinc (x) is an even function of x.
2. sinc $(x) = 0$ when $\sin x = 0$ except at $x = 0$, where it is indeterminate. This means that sinc $(x) = 0$ for $t = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
3. Using L'Hôpital's rule, we find sinc $(0) = 1$.
4. sinc (x) is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, sinc (x) exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
5. In summary, sinc (x) is an even oscillating function with decreasing amplitude. It has a unit peak at $x = 0$ and zero crossings at integer multiples of π.

Figure 3.9a shows sinc (x). Observe that sinc $(x) = 0$ for values of x that are positive and negative integral multiples of π. Figure 3.9b shows sinc $(3\omega/7)$. The argument $3\omega/7 = \pi$ when $\omega = 7\pi/3$ or $f = 7/6$. Therefore, the first zero of this function occurs at $\omega = 7\pi/3$ $(f = 7/6)$.

Example 3.2 Find the Fourier transform of $g(t) = \Pi(t/\tau)$ (Fig. 3.10a).

We have

$$G(f) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j2\pi f t}\, dt$$

Since $\Pi(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$G(f) = \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t}\, dt$$

$$= -\frac{1}{j2\pi f}(e^{-j\pi f \tau} - e^{j\pi f \tau}) = \frac{2 \sin(\pi f \tau)}{2\pi f}$$

$$= \tau \frac{\sin(\pi f \tau)}{(\pi f \tau)} = \tau \operatorname{sinc}(\pi f \tau)$$

Therefore,

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}\left(\frac{\omega \tau}{2}\right) = \tau \operatorname{sinc}(\pi f \tau) \tag{3.19}$$

Recall that $\operatorname{sinc}(x) = 0$ when $x = \pm n\pi$. Hence, $\operatorname{sinc}(\omega\tau/2) = 0$ when $\omega\tau/2 = \pm n\pi$; that is, when $f = \pm n/\tau$ $(n = 1, 2, 3, \ldots)$, as shown in Fig. 3.10b. Observe that in this case $G(f)$ happens to be real. Hence, we may convey the spectral information by a single plot of $G(f)$ shown in Fig. 3.10b, without having to show $|G(f)|$ and $\angle G(f)$.

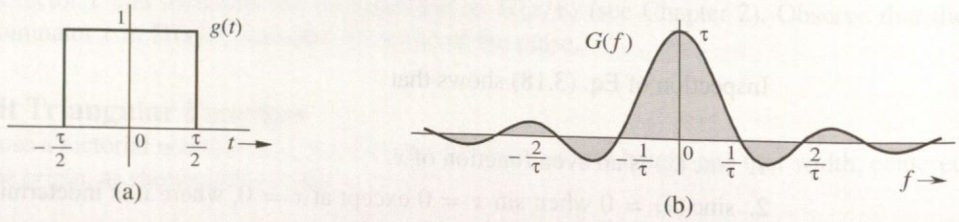


Figure 3.10 (a) Rectangular pulse and (b) its Fourier spectrum.

Bandwidth of $\Pi(\frac{t}{\tau})$

The spectrum $G(f)$ in Fig. 3.10 peaks at $f = 0$ and decays at higher frequencies. Therefore, $\Pi(t/\tau)$ is a low-pass signal with most of the signal energy in lower frequency components. **Signal bandwidth** is the difference between the highest (significant) frequency and the lowest (significant) frequency in the signal spectrum. Strictly speaking, because the spectrum extends from 0 to ∞, the bandwidth is ∞ in the present case. However, much of the spectrum is concentrated within the first lobe (from $f = 0$ to $f = 1/\tau$), and we may consider $f = 1/\tau$ to be the highest (significant) frequency in the spectrum. Therefore, a rough estimate of the bandwidth* of a rectangular pulse of width τ seconds is $2\pi/\tau$ rad/s, or $1/\tau$ Hz. Note the reciprocal relationship of the pulse width to its bandwidth. We shall observe later that this result is true in general.

* To compute the bandwidth, we must consider the spectrum for positive values of f only. The trigonometric spectrum exists only for positive frequencies. The negative frequencies occur because we use exponential spectra for mathematical convenience. Each sinusoid $\cos \omega_n t$ appears as a sum of two complex exponential components $e^{j\omega_n t}$ and $e^{-j\omega_n t}$ with frequencies ω_n and $-\omega_n$, respectively. But in reality, there is only one real component of frequency ω_n.

Example 3.3 Find the Fourier transform of the unit impulse signal $\delta(t)$.

We use the sampling property of the impulse function [Eq. (2.19)], to obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\,dt = e^{-j2\pi f\cdot 0} = 1 \tag{3.20a}$$

or

$$\delta(t) \Longleftrightarrow 1 \tag{3.20b}$$

Figure 3.11 shows $\delta(t)$ and its spectrum.

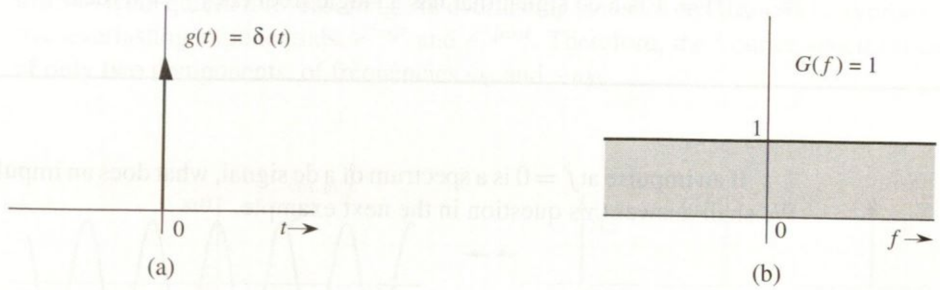


Figure 3.11 (a) Unit impulse and (b) its Fourier spectrum.

Example 3.4 Find the inverse Fourier transform of $\delta(2\pi f) = \frac{1}{2\pi}\delta(f)$.

From Eq. (3.9b) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(2\pi f)] = \int_{-\infty}^{\infty} \delta(2\pi f)e^{j2\pi ft}\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(2\pi f)e^{j2\pi ft}\,d(2\pi f)$$

$$= \frac{1}{2\pi}\cdot e^{-j0\cdot t} = \frac{1}{2\pi}$$

Therefore,

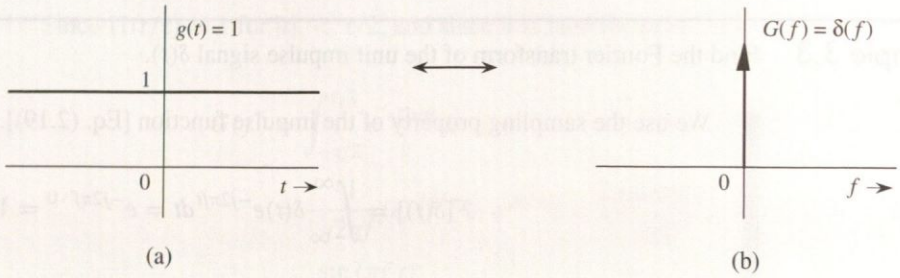
$$\frac{1}{2\pi} \Longleftrightarrow \delta(2\pi f) \tag{3.21a}$$

or

$$1 \Longleftrightarrow \delta(f) \tag{3.21b}$$

This shows that the spectrum of a constant signal $g(t) = 1$ is an impulse $\delta(f) = 2\pi\delta(2\pi f)$, as shown in Fig. 3.12.

Figure 3.12
(a) Constant (dc) signal and (b) its Fourier spectrum.



The result [Eq. (3.21b)] also could have been anticipated on qualitative grounds. Recall that the Fourier transform of $g(t)$ is a spectral representation of $g(t)$ in terms of everlasting exponential components of the form $e^{j2\pi ft}$. Now to represent a constant signal $g(t) = 1$, we need a single everlasting exponential $e^{j2\pi ft}$ with $f = 0$. This results in a spectrum at a single frequency $f = 0$. Another way of looking at the situation is that $g(t) = 1$ is a dc signal that has a single frequency $f = 0$ (dc).

If an impulse at $f = 0$ is a spectrum of a dc signal, what does an impulse at $f = f_0$ represent? We shall answer this question in the next example.

Example 3.5 Find the inverse Fourier transform of $\delta(f - f_0)$.

From the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(f - f_0)] = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi ft}\, df = e^{j2\pi f_0 t}$$

Therefore,

$$e^{j2\pi f_0 t} \Longleftrightarrow \delta(f - f_0) \tag{3.22a}$$

This result shows that the spectrum of an everlasting exponential $e^{j2\pi f_0 t}$ is a single impulse at $f = f_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting exponential $e^{j2\pi f_0 t}$, we need a single everlasting exponential $e^{j2\pi ft}$ with $f = f_0$. Therefore, the spectrum consists of a single component at frequency $f = f_0$.

From Eq. (3.22a) it follows that

$$e^{-j2\pi f_0 t} \Longleftrightarrow \delta(f + f_0) \tag{3.22b}$$

Example 3.6 Find the Fourier transforms of the everlasting sinusoid $\cos 2\pi f_0 t$.

Recall the Euler formula

$$\cos 2\pi f_0 t = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

Upon adding Eqs. (3.22a) and (3.22b), and using the preceding formula, we obtain

$$\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)] \tag{3.23}$$

The spectrum of $\cos 2\pi f_0 t$ consists of two impulses at f_0 and $-f_0$ in the f−domain, or, two impulses at $\pm\omega_0 = \pm 2\pi f_0$ in the ω−domain as shown in Fig. 3.13. The result also follows from qualitative reasoning. An everlasting sinusoid $\cos \omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore, the Fourier spectrum consists of only two components, of frequencies ω_0 and $-\omega_0$.

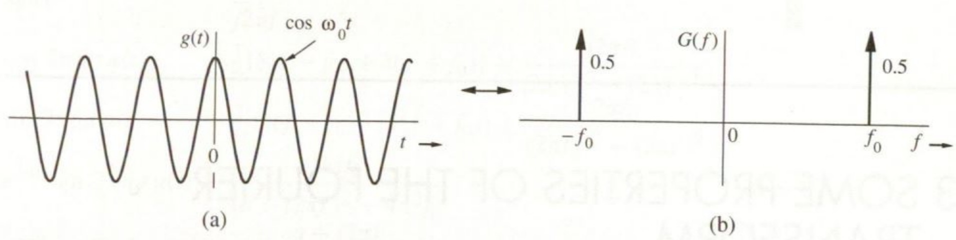


Figure 3.13 (a) Cosine signal and (b) its Fourier spectrum.

Example 3.7 Find the Fourier transform of the sign function sgn (t) (pronounced signum t), shown in Fig. 3.14. Its value is +1 or −1, depending on whether t is positive or negative:

$$\text{sgn}\,(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \tag{3.24}$$

We cannot use integration to find the transform of sgn (t) directly. This is because sgn (t) violates the Dirichlet condition (see Sec. 3.1). Specifically, sgn (t) is not absolutely integratable. However, the transform can be obtained by considering sgn t as a sum of two exponentials, as shown in Fig. 3.14, in the limit as $a \to 0$:

$$\text{sgn}\, t = \lim_{a \to 0} \left[e^{-at}u(t) - e^{at}u(-t)\right]$$

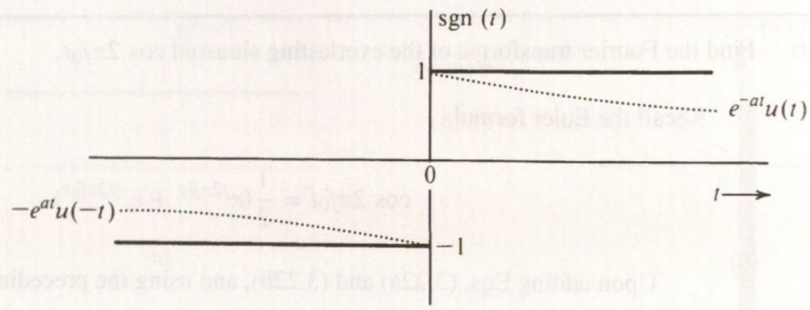


Figure 3.14
Sign function.

Therefore,

$$\begin{aligned}
\mathcal{F}\,[\mathrm{sgn}(t)] &= \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\} \\
&= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) \qquad \text{(see pairs 1 and 2 in Table 3.1)} \\
&= \lim_{a\to 0}\left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f}
\end{aligned} \tag{3.25}$$

3.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

We now study some of the important properties of the Fourier transform and their implications as well as their applications. Before embarking on this study, it is important to point out a pervasive aspect of the Fourier transform—the **time-frequency duality**.

3.3.1 Time-Frequency Duality

Equations (3.9a) and (3.9b) show an interesting fact: the direct and the inverse transform operations are remarkably similar. These operations, required to go from $g(t)$ to $G(f)$ and then from $G(f)$ to $g(t)$, are shown graphically in Fig. 3.15. The only minor difference between these two operations lies in the opposite signs used in their exponential functions.

This similarity has far-reaching consequences in the study of the Fourier transform. It is the basis of the so-called duality of time and frequency. *The duality principle may be considered by analogy to a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, a negative can be obtained from the photograph.* For any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of a sign change). For example, the time-shifting property, to be proved later, states that if $g(t) \Longleftrightarrow G(f)$, then

$$g(t-t_0) \Longleftrightarrow G(f)e^{-j2\pi f t_0}$$